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Rate Matrix of G/M/1-type Markov Chain

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Abstract. In this short note we consider the matrix-geometric solution of the so-called G/M/1-type structured Markov chain and obtain a few results related to the closed-form expression for the so-called rate matrix. In particular, it is shown that the rate matrix is a solution of the monic matrix polynomial equation. This result may be useful in obtaining the matrix explicitly, which we demonstrate by studying an interesting batch-service model.

Keywords: Matrix Polynomial · G/M/1-type Markov Chain · Matrix-Analytic Method.

1 Introduction

Consider a structured, skip-free to the right, also known as $G/M/1$ -type, continuous-time Markov chain (CTMC)

$$\{X(t), Y(t)\}_{t \geq 0}, \quad (1)$$

and let $\mathbf{A}^{(i)}$, $i = 0, 1, \dots, N$ be square $n \times n$ (finite size) blocks of its quasi-Toeplitz generator matrix,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}^{0,0} & \mathbf{A}^{0,1} & \mathbb{O} & \mathbb{O} & \dots \\ \mathbf{A}^{1,0} & \mathbf{A}^{(1)} & \mathbf{A}^{(0)} & \mathbb{O} & \dots \\ \mathbf{A}^{2,0} & \mathbf{A}^{(2)} & \mathbf{A}^{(1)} & \mathbf{A}^{(0)} & \dots \\ \mathbf{A}^{3,0} & \mathbf{A}^{(3)} & \mathbf{A}^{(2)} & \mathbf{A}^{(1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \mathbf{A}^{(N)} & \mathbf{A}^{(N-1)} & \mathbf{A}^{(N-2)} & \dots \\ \vdots & \mathbb{O} & \mathbf{A}^{(N)} & \mathbf{A}^{(N-1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2)$$

where $\mathbf{A}^{i,0}$, $i > 0$, and $\mathbf{A}^{0,1}$ are non-negative (possibly rectangular) matrices and square matrix $\mathbf{A}^{0,0}$ has negative diagonal and non-negative off-diagonal elements. We consider the case $N \geq 2$ with $N = 2$ corresponding to the so-called

Quasi-Birth-Death process. We formally allow $N = \infty$, which corresponds to the classical G/M/1-type Markov chain unless finiteness of N is stated explicitly. Note that $\mathbf{A}^{(i)} \geq \mathbb{O}$ for $i \neq 1$ and $\mathbf{A}^{(1)}$ has negative diagonal elements, non-negative off-diagonal elements s.t. $\mathbf{Q}\mathbf{1} = \mathbf{0}$ (we use bold letters for vectors and matrices, $\mathbf{1}$ is the vector of ones, $\mathbf{0}$ is the vector of zeroes and $\mathbb{O}$ is the zero matrix).

Consider a matrix (generating) function $P(x)$ defined (formally) for x being either a matrix or a scalar

$$P(x) := \sum_{i=0}^N x^i \mathbf{A}^{(i)}. \quad (3)$$

For $N < \infty$, $P(x)$ defines a (matrix) polynomial of x . Define

$$\mathbf{A} = P(1) = \sum_{i=0}^N \mathbf{A}^{(i)}.$$

In case $N = \infty$ we consider the corresponding series converge when needed so (in particular, for scalar $x \in [0, 1]$), which may be guaranteed, e.g., by the fact that $\mathbf{A}$ is a generator matrix, but we do not elaborate on this issue and refer the reader to e.g. [9, Section 1.3].

We assume the steady state distribution $\boldsymbol{\pi}$ of the CTMC (1) exists, which is subject to the following criterion [3, Condition 8.33],

$$\boldsymbol{\alpha} \sum_{k=1}^{N-1} k \mathbf{A}^{(k+1)} \mathbf{1} > \boldsymbol{\alpha} \mathbf{A}^{(0)} \mathbf{1}, \quad (4)$$

where $\boldsymbol{\alpha}$ is the invariant stochastic vector of $\mathbf{A}$,

$$\boldsymbol{\alpha} \mathbf{A} = \mathbf{0}, \quad \boldsymbol{\alpha} \mathbf{1} = 1.$$

The spectrally minimal, componentwise nonnegative solution matrix $\mathbf{R}$ of

$$P(\mathbf{R}) = \mathbb{O} \quad (5)$$

is used to obtain $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \dots)$ in a block-vector form [3, Theorem 8.3]:

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0 \mathbf{R}^i, \quad i = 1, 2, \dots$$

Take some real $r \in (-1, 1)$ and observe that

$$P(r) = (r\mathbf{I} - \mathbf{R})E(r, \mathbf{R}), \quad (6)$$

where

$$E(r, \mathbf{R}) := \sum_{i=0}^{N-1} r^i \mathbf{E}_i, \quad (7)$$

and, for $i = 0, \dots, N-1$,

$$\mathbf{E}_i = \sum_{j=i+1}^N \mathbf{R}^{j-i-1} \mathbf{A}^{(j)}. \quad (8)$$

Note that one can check from (8) that

$$\mathbf{E}_i - \mathbf{R}\mathbf{E}_{i+1} = \mathbf{A}^{(i+1)}, \quad i = 0, \dots, N-2.$$

Factorization (6), formally stated for $N < \infty$ in [7], is also known for certain types of matrix polynomials as the left divisibility of $P(r)$, see e.g. [5, Sec. 4.2]. For $N = \infty$, using (5), one has

$$P(r) = \sum_{i=0}^{\infty} r^i \mathbf{A}^{(i)} = \sum_{i=0}^{\infty} (r^i \mathbf{I} - \mathbf{R}^i) \mathbf{A}^{(i)} = \sum_{i=1}^{\infty} (r\mathbf{I} - \mathbf{R}) \left[\sum_{j=0}^{i-1} r^j \mathbf{R}^{i-1-j} \right] \mathbf{A}^{(i)}.$$

Then (6) follows by changing the order of summation, where $E(r, \mathbf{R})$ in (7) and $\mathbf{E}_i$ from (8) are series. It follows from (6) that the eigenvalues of $\mathbf{R}$ are those n solutions (counting their multiplicity) of the equation

$$\det P(x) = 0, \quad (9)$$

that are within the unit disk.

In what follows, we are interested in a closed form expression for the matrix $\mathbf{R}$. In Section 2 we obtain an interesting result related to the spectre of matrix $\mathbf{R}$ for finite N , whereas in Section 3 the closed-form expression for $\mathbf{R}$ is obtained by using Cayley–Hamilton theorem. We complete the paper with an example of a batch-service system in Section 4.

2 Finite N

One can consider right divisibility of $P(r)$,

$$P(r) = K(r, \mathbf{F})(\mathbf{I} - r\mathbf{F}), \quad (10)$$

where

$$K(r, \mathbf{F}) = \sum_{i=0}^{N-1} r^i \mathbf{K}_i,$$

and one can check by direct inspection that

$$\mathbf{K}_i = \sum_{j=0}^i \mathbf{A}^{(i-j)} \mathbf{F}^j, \quad i = 0, \dots, N-1, \quad (11)$$

where $\mathbf{F}$ is a stochastic matrix which is the solution of

$$\sum_{i=0}^N \mathbf{A}^{(i)} \mathbf{F}^{N-i} = \mathbf{O}. \quad (12)$$

It follows from (11) that

$$\mathbf{K}_{i+1} - \mathbf{K}_i \mathbf{F} = \mathbf{A}^{(i+1)}, \quad i = 0, \dots, N-1.$$

One would expect from (6) and (10) that $K(r, \mathbf{F})$ is left divisible by $r\mathbf{I} - \mathbf{R}$. Indeed, consider the equation

$$(r\mathbf{I} - \mathbf{R}) \left(\sum_{i=0}^{N-1} r^i \mathbf{T}_i \right) = \sum_{i=0}^N r^i \mathbf{K}_i, \quad (13)$$

for some unknown $\mathbf{T}_0, \dots, \mathbf{T}_{N-1}$. Rewriting l.h.s. of (13) as

$$-\mathbf{R}\mathbf{T}_0 + \left[\sum_{i=1}^{N-1} r^i (\mathbf{T}_{i-1} - \mathbf{R}\mathbf{T}_i) \right] + r^N \mathbf{T}_{N-1},$$

one has

$$\mathbf{K}_N = \mathbf{T}_{N-1} \quad (14)$$

$$\mathbf{K}_i = \mathbf{T}_{i-1} - \mathbf{R}\mathbf{T}_i, \quad i = 1, \dots, N-1, \quad (15)$$

$$\mathbf{K}_0 = -\mathbf{R}\mathbf{T}_0. \quad (16)$$

These equations, after rewriting into recursion starting from $\mathbf{T}_{N-1}$, give the coefficients of the needed factorization as follows,

$$\mathbf{T}_{N-1} = \mathbf{K}_N \quad (17)$$

$$\mathbf{T}_{i-1} = \mathbf{K}_i + \mathbf{R}\mathbf{T}_i, \quad i = N-1, \dots, 1, \quad (18)$$

which gives in a closed form

$$\mathbf{T}_{N-i} = \sum_{j=0}^{i-1} \mathbf{R}^j \mathbf{K}_{N-i+1+j}, \quad i = 1, \dots, N. \quad (19)$$

Thus, finally one has

$$P(r) = (r\mathbf{I} - \mathbf{R}) \left(\sum_{i=0}^{N-1} r^i \mathbf{T}_i \right) (\mathbf{I} - r\mathbf{F}).$$

Here the matrix $\mathbf{R}$ has the eigenvalues inside unit disk, and $\mathbf{F}$ as well. As a result, $\det(\mathbf{I} - r\mathbf{F})$ has roots outside the unit disk. This result is a generalization of [8, Theorem 7.5], see also [2, Theorem 3.20], and in fact is a variant of Wiener–Hopf factorization, see also [1,10].

3 Case $\mathbf{R}$ is diagonalizable

In this section we assume $\mathbf{R}$ is diagonalizable. Consider the characteristic equation for the eigenvalues of $\mathbf{R}$,

$$\det(r\mathbf{I} - \mathbf{R}) = 0. \quad (20)$$

It is known [6, Theorem 3.4.3] that, given (4) holds, $\text{sp}(\mathbf{R}) < 1$, i.e. all the solutions of (20) are within the unit disk. Denoting the l.h.s of (20) by $C(r)$, we can rewrite it into the n -th power polynomial of r ,

$$C(r) = r^n + b_{n-1}r^{n-1} + \cdots + b_1r + b_0 = 0. \quad (21)$$

By the Cayley–Hamilton theorem, $C(\mathbf{R}) = \mathbb{O}$, which gives

$$\mathbf{R}^n = -\sum_{i=0}^{n-1} b_i \mathbf{R}^i, \quad (22)$$

or, in matrix form,

$$\mathbf{R}^n = \sum_{i=1}^n \mathbf{e}_i \mathbf{C}_n \mathbf{e}_1' \mathbf{R}^{n-i}, \quad (23)$$

where $\mathbf{e}_i$ is the i -th n -dimensional ort, $\mathbf{C}_n$ is the companion matrix:

$$\mathbf{C}_n = \begin{bmatrix} -b_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -b_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots \\ -b_1 & 0 & 0 & 0 & \cdots & 1 \\ -b_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = -\tilde{\mathbf{b}}' \mathbf{e}_1 + \mathbf{U}, \quad (24)$$

where $\tilde{\mathbf{b}} = (b_{n-1}, \dots, b_0)$ and $\mathbf{U}$ is the matrix with unit upper diagonal. On noting that $\mathbf{e}_i \mathbf{e}_j' = \delta(i, j)$, that $\mathbf{e}_n \mathbf{U} = \mathbf{U} \mathbf{e}_1' = \mathbf{0}$, also

$$\mathbf{e}_i \tilde{\mathbf{b}}' = \tilde{\mathbf{b}} \mathbf{e}_i' = b_{n-i}, \quad (25)$$

and finally

$$\mathbf{U} \mathbf{e}_i' = \mathbf{e}_{i-1}', \quad \mathbf{e}_i \mathbf{U} = \mathbf{e}_{i+1}, \quad i \geq 2,$$

one can derive from (24) the following useful properties of the companion matrix,

$$\mathbf{C}_n \mathbf{e}_i' = \mathbf{e}_{i-1}', \quad i \geq 2, \quad (26)$$

$$\mathbf{C}_n \mathbf{e}_1' = -\tilde{\mathbf{b}}', \quad (27)$$

$$\mathbf{e}_i \mathbf{C}_n = -b_{n-i} \mathbf{e}_1 + \mathbf{e}_{i+1}, \quad i < n, \quad (28)$$

$$\mathbf{e}_n \mathbf{C}_n = -b_0 \mathbf{e}_1. \quad (29)$$

Multiply both sides of (22) by $\mathbf{R}$ to get the following recursive equation

$$\begin{aligned}
\mathbf{R}^{n+1} &= \mathbf{e}_1 \mathbf{C}_n \mathbf{e}'_1 \mathbf{R}^n + \sum_{i=2}^n \mathbf{e}_i \mathbf{C}_n \mathbf{e}'_1 \mathbf{R}^{n-i+1} \\
&= \mathbf{e}_1 \mathbf{C}_n \mathbf{e}'_1 \sum_{i=1}^n \mathbf{e}_i \mathbf{C}_n \mathbf{e}'_1 \mathbf{R}^{n-i} + \sum_{i=1}^{n-1} \mathbf{e}_{i+1} \mathbf{C}_n \mathbf{e}'_1 \mathbf{R}^{n-i} \\
&= \sum_{i=1}^{n-1} (\mathbf{e}_1 \mathbf{C}_n \mathbf{e}'_1 \mathbf{e}_i + \mathbf{e}_{i+1}) \mathbf{C}_n \mathbf{e}'_1 \mathbf{R}^{n-i} + \mathbf{e}_1 \mathbf{C}_n \mathbf{e}'_1 \mathbf{e}_n \mathbf{C}_n \mathbf{e}'_1 \mathbf{I} \\
&= \sum_{i=1}^n \mathbf{e}_i \mathbf{C}_n^2 \mathbf{e}'_1 \mathbf{R}^{n-i},
\end{aligned}$$

where the last assertion follows from (25), (27), (28) and (29). Indeed, since $\mathbf{e}_i \mathbf{C}_n \mathbf{e}'_1$ is a scalar for $i = 1, \dots, n$, a direct inspection using (28) gives

$$\mathbf{e}_1 \mathbf{C}_n \mathbf{e}'_1 \mathbf{e}_i \mathbf{C}_n \mathbf{e}'_1 + \mathbf{e}_{i+1} \mathbf{C}_n \mathbf{e}'_1 = (\mathbf{e}_i \mathbf{C}_n \mathbf{e}'_1 \mathbf{e}_1 + \mathbf{e}_{i+1}) \mathbf{C}_n \mathbf{e}'_1 = \mathbf{e}_i \mathbf{C}_n^2 \mathbf{e}'_1,$$

whereas

$$\mathbf{e}_1 \mathbf{C}_n \mathbf{e}'_1 \mathbf{e}_n \mathbf{C}_n \mathbf{e}'_1 = -b_0 \mathbf{e}_1 \mathbf{C}_n \mathbf{e}'_1 = \mathbf{e}_n \mathbf{C}_n^2 \mathbf{e}'_1.$$

By induction, one has the final form

$$\mathbf{R}^{n+k} = \sum_{i=1}^n \mathbf{e}_i \mathbf{C}_n^{k+1} \mathbf{e}'_1 \mathbf{R}^{n-i}, \quad k \geq 0. \quad (30)$$

Note that (30) is a matrix form of the recursion obtained in [4] for $n = 2$ and further developed in [11].

Using (30), one can formally rewrite (5) to obtain $\mathbf{R}$ as the solution of the following equation,

$$\sum_{i=0}^{n-1} \mathbf{R}^i \mathbf{A}^{(i)} + \sum_{i=1}^n \mathbf{e}_i \left[\sum_{k=0}^N \mathbf{C}_n^{k+1} \mathbf{e}'_1 \mathbf{R}^{n-i} \mathbf{A}^{(n+k)} \right] = \mathbb{O}, \quad (31)$$

which is matrix polynomial equation in $\mathbf{R}$ both in case N is finite or infinite. On noting that $\mathbf{e}_i \mathbf{C}_n^{k+1} \mathbf{e}'_1$ is scalar, making a change in the second summation argument $i := n - i$, one may rewrite (31) in a slightly modified form

$$\sum_{i=0}^{n-1} \mathbf{R}^i \left[\mathbf{A}^{(i)} + \sum_{k=0}^N \mathbf{e}_{n-i} \mathbf{C}_n^{k+1} \mathbf{e}'_1 \mathbf{A}^{(n+k)} \right] = \mathbb{O}. \quad (32)$$

In particular, (32) is linear if $n = 2$, and one may check that for $N < \infty$ the linear equation (32) coincides with the one obtained in [4], and the matrix $\mathbf{R}$ can be obtained in explicit form along the lines of [4]. Below we demonstrate this by studying an interesting batch-service system.

4 Example: Batch Service System

We consider a two-class batch-service system. Customers arrivals are Poisson with rate λ , the customer is class-1 with probability (w.p.) p_1 and class-2 w.p. $p_2 = 1 - p_1$. Service times are exponentially distributed with a class-dependent rate μ_i . During a single service time, a sequential batch of customers having the same class as the oldest customer in the system, including those who might have arrived during the ongoing service, is served. Denote $X(t)$ the number of customers in the system and $Y(t)$ the class that is being served at time $t \geq 0$.

Fix $X(t) = n$ and note that the arrivals do not change $Y(t)$. On the contrast, upon departure, $Y(t)$ is alternated if $X(t) > 0$ after departure. Indeed, in such a case the head-of-queue customer is not of class $Y(t)$, and the next batch taken to service is of the alternative class, $3 - Y(t)$. Due to the iid nature of the classes arriving into system, the queue contents has Bernoulli distribution. As such, the server takes $n \geq 1$ customers to service w.p. $p_{Y(t)}^{n-1}$, and in such a case, the departure leaves the system empty. Otherwise $k \in 1, \dots, n-1$ customers are served w.p. $p_{Y(t)}^{k-1}(1 - p_{Y(t)})$, $n - k$ customers are left in the queue (if $n > 1$), and $Y(t)$ is changed into alternate class, $3 - Y(t)$ (i.e. server switches from class 1 to class 2 and vice versa). Thus, the generator of the G/M/1-type process $\{X(t), Y(t)\}_{t \geq 0}$ is of the form (2), where the block that corresponds to arrivals, balance block and the blocks corresponding to $i - 1$ departures that do not leave the system empty (note $N = \infty$), are, respectively

$$\mathbf{A}^{(0)} = \lambda \mathbf{I}, \quad \mathbf{A}^{(1)} = - \begin{bmatrix} \lambda + \mu_1 & 0 \\ 0 & \lambda + \mu_2 \end{bmatrix}, \quad (33)$$

$$\mathbf{A}^{(i)} = \begin{bmatrix} 0 & p_1^{i-2} p_2 \mu_1 \\ p_2^{i-2} p_1 \mu_2 & 0 \end{bmatrix}, \quad i \geq 2. \quad (34)$$

The block that corresponds to arrival into empty system, the balance block for empty system, and the blocks that correspond to i departures leaving the system empty, are

$$\mathbf{A}^{0,1} = [\lambda p_1 \quad \lambda p_2], \quad \mathbf{A}^{0,0} = -\lambda, \quad \mathbf{A}^{i,0} = \begin{bmatrix} p_1^{i-1} \mu_1 \\ p_2^{i-1} \mu_2 \end{bmatrix}, \quad i \geq 1. \quad (35)$$

One can check from (4) using (33) and (34), after some straightforward algebra, that the stability condition of the system is

$$\lambda < \left(\frac{1}{p_1} + \frac{1}{p_2} \right) \left[\frac{1}{\mu_1} + \frac{1}{\mu_2} \right]^{-1}.$$

It has a clear intuitive interpretation: the input rate is smaller than the average service rate of two average consecutive batches of class-1 and class-2 customers. Indeed, at high queue levels, class-1 and class-2 batches are served one after another in an alternating manner, whereas each such a batch size is geometrically distributed.

Let us solve (9), where $P(x)$ is defined in (3). It can be shown that

$$P(x) = \begin{bmatrix} \lambda - (\lambda + \mu_1)x & \mu_1 p_2 \sum_{i=2}^{\infty} x^i p_1^{i-2} \\ \mu_2 p_1 \sum_{i=2}^{\infty} x^i p_2^{i-2} & \lambda - (\lambda + \mu_2)x \end{bmatrix}.$$

As such,

$$\det P(x) = (\lambda - (\lambda + \mu_1)x)(\lambda - (\lambda + \mu_2)x) - \frac{\mu_1 \mu_2 p_1 p_2 x^4}{(1 - x p_1)(1 - x p_2)}.$$

It is easy to check that $x = 1$ is the solution of (9). Avoiding singularities of the denominator and dividing by $(1 - x)$ gives the third order polynomial $P_1(x) = c_3 x^3 + \dots + c_0$ where

$$\begin{aligned} c_3 &= -\lambda(\lambda + \mu_1 + \mu_2)p_1 p_2, \quad c_2 = \lambda^2(1 + p_1 p_2) + \lambda(\mu_1 + \mu_2) + \mu_1 \mu_2, \\ c_1 &= -\lambda(2\lambda + \mu_1 + \mu_2), \quad c_0 = \lambda^2. \end{aligned}$$

One can obtain the (absolute) largest root x_3 of this cubic equation in trigonometric form as

$$x_3 = -c_2/(3c_3) + 2\sqrt{-p} \cos(\arccos(q/(2p\sqrt{-p}))/3), \quad (36)$$

where

$$p = \frac{3c_3 c_1 - c_2^2}{9c_3^2}, \quad q = \frac{2c_2^3 - 9c_3 c_2 c_1 + 27c_3^2 c_0}{27c_3^3}.$$

Using x_3 , and denoting the roots within the unit disk as x_1, x_2 , one can define the values of the coefficients of the characteristic polynomial

$$b_1 = -(x_1 + x_2) = \frac{c_2}{c_3} + x_3, \quad b_0 = x_1 x_2 = -\frac{c_0}{c_3 x_3}.$$

Adopting (32) for $n = 2$ one has, after some straightforward algebra

$$\mathbf{R} = -\mathbf{M}_2 \mathbf{M}_1^{-1}, \quad (37)$$

where

$$\mathbf{M}_1 = \begin{bmatrix} -\lambda - \mu_1 & \sigma(p_1, \mu_1) \\ \sigma(p_2, \mu_2) & -\lambda - \mu_2 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} \lambda & s(p_1, \mu_1) \\ s(p_2, \mu_2) & \lambda \end{bmatrix}, \quad (38)$$

and

$$\sigma(y, \mu) = \frac{-b_1(1 - y)\mu - b_0 y(1 - y)\mu}{1 + b_1 y + b_0 y^2}, \quad s(y, \mu) = -\frac{b_0(1 - y)\mu}{1 + b_1 y + b_0 y^2}.$$

The key steps in computing $\mathbf{M}_1$ and $\mathbf{M}_2$ can be done by using the eigendecomposition of $\mathbf{C}_2$,

$$\mathbf{C}_2 = x_1 \mathbf{\Delta}_1 + x_2 \mathbf{\Delta}_2, \quad (39)$$

where the idempotent matrices Δ_i , $i = 1, 2$ are as follows,

$$\Delta_1 = \begin{bmatrix} \frac{x_1}{x_1 - x_2} & \frac{1}{x_1 - x_2} \\ -\frac{x_1 x_2}{x_1 - x_2} & -\frac{x_2}{x_1 - x_2} \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} \frac{x_2}{x_2 - x_1} & \frac{1}{x_2 - x_1} \\ -\frac{x_1 x_2}{x_2 - x_1} & -\frac{x_1}{x_2 - x_1} \end{bmatrix}.$$

As such, one has from (39) that

$$C_2^k = x_1^k \Delta_1 + x_2^k \Delta_2, \quad k \geq 0.$$

In order to use (32), one can check that

$$e_1 C_2^k e'_1 = \frac{x_1^{k+1} - x_2^{k+1}}{x_1 - x_2} = \sum_{i=0}^k x_1^i x_2^{k-i}, \quad k \geq 0, \quad (40)$$

$$e_2 C_2^k e'_1 = -x_1 x_2 \frac{x_1^k - x_2^k}{x_1 - x_2} = -\sum_{i=0}^{k-1} x_1^{i+1} x_2^{k-i}, \quad k \geq 1. \quad (41)$$

These coefficients coincide with those obtained in [4, Theorem 1] for finite N . Finally, in order to derive (37), one can use (40) and (41) in (32) and change the order of summation, e.g.

$$\begin{aligned} s(p_1, \mu_1) &= \sum_{k=0}^{\infty} e_2 C_2^{k+1} e'_1 p_1^k p_2 \mu_1 = -\sum_{k=0}^{\infty} \sum_{i=0}^k x_1^{i+1} x_2^{k+1-i} p_1^k p_2 \mu_1 \\ &= -x_1 x_2 p_2 \mu_1 \sum_{i=0}^{\infty} x_1^i p_1^i \sum_{k=i}^{\infty} x_2^{k-i} p_1^{k-i} = -\frac{b_0(1-p_1)\mu_1}{1+b_1 p_1 + b_0 p_1^2}. \end{aligned}$$

Note that the final expressions in (38) do not use the values x_1 and x_2 , but b_1 and b_0 are used instead. Thus, only computation of x_3 is needed, as done by (36).

In order to obtain the stationary distribution $\pi = (\pi_0, \pi_1, \dots)$ (in a level-wise form), on noting that $\pi_k = \pi_1 R^{k-1}$, $k \geq 1$, it remains to solve the boundary system of equations

$$(\pi_0, \pi_1) \left[\sum_{i=1}^{\infty} R^{i-1} A^{i,0} \quad \sum_{i=1}^{\infty} R^{i-1} A^{(i)} \right] = \mathbf{0}, \quad (42)$$

under the normalizing condition $\pi \mathbf{1} = 1$, which gives an equation

$$\pi_0 + \pi_1 (I - R)^{-1} \mathbf{1} = 1.$$

Using the latter equation instead of the first in (42), and noting that

$$\sum_{i=1}^{\infty} R^{i-1} A^{(i)} = -R^{-1} A^{(0)} = -\lambda R^{-1},$$

we get the final form

$$(\pi_0, \pi_1) \begin{bmatrix} 1 & \mathbf{A}^{0,1} \\ (\mathbf{I} - \mathbf{R})^{-1} \mathbf{1} & -\lambda \mathbf{R}^{-1} \end{bmatrix} = \mathbf{e}_1,$$

which gives the required values of π_0 and π_1 . Using that, one can obtain, say, the stationary system contents $E\nu$ in explicit form

$$E\nu = \pi_1 (\mathbf{I} - \mathbf{R})^{-2} \mathbf{1}.$$

For an illustrative purpose, we depict the dependence of $E\nu$ on λ in Figure 1.

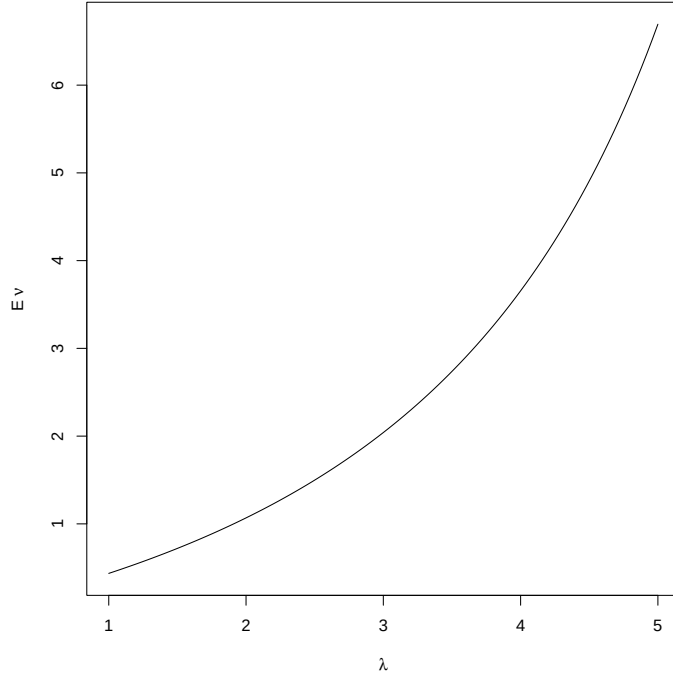


Fig. 1. Dependence of the average stationary system contents $E\nu$ on the input rate $\lambda \in [1, 5]$, for $p_1 = 0.2$, $\mu_1 = 2$ and $\mu_2 = 3$ fixed.

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