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Asymptotic Analysis of Multi-thread Heterogeneous Queuing System under Conditions of Extremely Rare Changes in the States of the Markov Chain Controlling Input Flows

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Abstract. The paper examines a multi-threaded queuing system where flow intensity changes according to the states of a Markov random environment. Incoming requests from various flows are served during an exponentially distributed random time with parameters set by the type of flow. The random environment does not influence service. This study is conducted under the assumption of extremely rare changes in environmental conditions. The form of the multi-dimensional asymptotic characteristic function is established. It is proved that one-dimensional (marginal) stationary probability distributions of the number of engaged devices of each type are weighted sums of Poisson distributions. A numerical analysis was performed to determine the range of applicability of the established approximation.

Keywords: Markov modulated Poisson flows· asymptotic analysis· extremely rare changes in the states of a Markov chain

1 Introduction

Today we are surrounded by many various application areas that require the transmission of multimodal data. Examples of such areas include biomedical applications (intensive care monitoring and medical imaging) [1], transportation systems (smart car and traffic systems) [2], multimedia information analysis (audiovisual human identification, multimodal robot interaction and multimodal video search) [3].

In this context, modality refers to physically registered components of communication, encompassing both human-machine and interpersonal interactions. It includes not only the actual transmitted information, but also data about the individual, such as their condition and attitudes towards the message, the conversation, and communication in general [4].

Considering that multimodal flows integrate various types of data such as voice, text data, and video, it is logical to use non-Poisson models to describe them. The time taken to serve different information units varies, depending on the format, protocols used, and other parameters. As such, queuing systems with diverse types of requests, each requiring different service times, are used to model processes in information systems.

Currently, the solution for analyzing non-Markov multidimensional queuing models with non-Poisson incoming flows and non-exponential services is limited to specific works. These works typically focus on models of a narrow class or a specific configuration [5–9]. Thus, developing fundamental approaches and methods to study non-Markovian queuing models is a pressing scientific issue.

This paper presents a model for a multi-threaded, heterogeneous data transmission system. It operates over channels with different service intensities, taking the form of a heterogeneous queuing system.

2 Mathematical Model

In this paper we consider a queuing system $MMPP^{(n)}|M_n|\infty$ This is an infinite-server queuing system with various types of MMPP flows arriving at the input and corresponding diverse types of services – each differing in their service speed.

The input receives N Poisson flows whose intensities are determined by the state of the random environment. This environment is a continuous-time Markov chain with a state number $k(t)$ that ranges from 1 to K. The generator matrix determines the chain's transition rates $\mathbf{Q} = [q_{ij}]$ $(i, j = 1, \dots K)$.

Therefore, we can conclude that the incoming flows are Markov modulated Poisson flows [10,11]. These flows have a common controlled process $k(t)$ and are determined by the diagonal matrices of conditional intensities Λ^l $(l = 1, ..., N)$ with elements $\lambda_k^{(l)}$ on the main diagonal.

An incoming request always finds a free service block and occupies it for service for a random time distributed according to an exponential law with parameters μ_l $(l = 1, \dots, N)$, respectively. Thus, the system contains requirements of different types and different service parameters, which we will conventionally define as different blocks. The problem is set to study an N-dimensional non-Markov random process that describes the number of requests of each type in the corresponding service block.

To construct a Markov random process, we will use the supplementary variable technique [7]. Consider the state of the random environment, $k(t)$. Construct the $N+1$-dimensional Markov process $\{k(t), i_1(t), i_2(t), \dots, i_N(t)\}$. Our objective is to define the joint probability distribution for this process,

$$P = (k, i_1, i_2, \dots, i_N, t) = P\{k(t) = k, i_1(t) = i_1, i_2(t) = i_2, \dots, i_N(t) = i_N\}.$$

We use the vector representation for a more compact notation. To do this, we demote the following notation: $\mathbf{i(t)} = [i_1(t), i_2(t), \dots, i_N(t)]$, $\mathbf{e}_1 = [1, 0, \dots, 0]$, $\mathbf{e}_2 = [0, 1, \dots, 0], \mathbf{e}_N = [0, 0, \dots, 1]$. These are $1 \times N$ row vectors.

For stationary probabilities $\Pi(k, \mathbf{i})$, $k = 1, \ldots, K$, $\mathbf{i} \geq \mathbf{0}$, the Kolmogorov system of equations has the following form (with obvious changes if $\mathbf{i} < \mathbf{e}_l$):

$$\begin{aligned} 0 = -\Pi(k, \mathbf{i}) \left[\sum_{l=1}^{N} (\lambda_k^{(l)} + i_l \mu_l) \right] + \sum_{l=1}^{N} \lambda_k^{(l)} \Pi(k, \mathbf{i} - \mathbf{e}_l) + \\ + \sum_{l=1}^{N} (i_l + 1) \mu_l \Pi(k, \mathbf{i} + \mathbf{e}_l)) + \sum_{v=1}^{K} q_{vk} \Pi(k, \mathbf{i}). \end{aligned} \tag{1}$$

For $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ and $k = 1, \ldots, K$ let us introduce the partial characteristic functions:

$$h(k, \mathbf{u}) = \sum_{i_1} e^{ju_1 i_1} \sum_{i_2} e^{ju_2 i_2} \cdots \sum_{i_l} e^{ju_S i_N} \Pi(k, \mathbf{i}),$$

where $j = \sqrt{-1}$ is the imaginary unit.

For these characteristics the system of equations can be rewritten as follows:

$$\begin{gathered} j \sum_{l=1}^{N} \mu_l (e^{-ju_l} - 1) \frac{\partial h(k, \mathbf{u})}{\partial u_l} = \sum_{l=1}^{N} \lambda_k^{(l)} (e^{ju_l} - 1) h(k, \mathbf{u}) + \sum_{v=1}^{K} q_{vk} h(v, \mathbf{u}), \\ k = 1, \ldots, K, \end{gathered} \tag{2}$$

under the initial conditions $h(k, \mathbf{0}) = r(k)$, where $\mathbf{r} = [r(1), r(2), ..., r(K)]$ is the stochastic row vector of the stationary probability distribution of states of the random environment determined by a system of linear equations

$$\mathbf{r}\mathbf{Q} = 0.$$

3 Asymptotic Analysis under the Condition of Extremely Rare Changes in the Random Environment

Here we present some results for a solution to the system under the condition of extremely rare changes in the states of the random environment. It is known [7] that the values of the diagonal elements of the generator matrix determine the duration the flow remains in the corresponding state. Denoting T_k as the duration of the Markov chain k-th state, we obtain that

$$T_k = -\frac{1}{q_{kk}}, k = 1, \ldots, K.$$

If q_{kk} takes sufficiently small values, then the sojourn time in the k-th state increases without limit, which justifies the name of the asymptotic condition considered. Therefore, the following equality can be used to describe the condition for extremely rare changes in the states of the Markov chain:

$$q_{vk} = \varepsilon q_{vk}^*, \quad \varepsilon \to 0, \quad \text{where } \varepsilon \text{ is some small positive parameter.} \tag{3}$$

We rewrite the system of equations (2) in the following form, considering (3):

$$j\sum_{l=1}^{N}\mu_l(e^{-ju_l}-1)\frac{\partial h(k,\mathbf{u},\varepsilon)}{\partial u_l}= \\ =\sum_{l=1}^{N}\lambda_k^{(l)}(e^{ju_l}-1)h(k,\mathbf{u},\varepsilon)+\varepsilon\sum_{v=1}^{K}q^*_{vk}h(v,\mathbf{u},\varepsilon). \tag{4}$$

The theorem on the continuous dependency of the system of differential equations solution on the parameter states that there exists a limit

$$\lim_{\varepsilon\to 0}h(k,\mathbf{u},\varepsilon)=h_1(k,\mathbf{u}).$$

Next, we have a system of partial differential equations of the following form by taking system (4) to the limit as $\varepsilon\to 0$.

$$j\sum_{l=1}^{N}\mu_l(e^{-ju_l}-1)\frac{\partial h_1(k,\mathbf{u})}{\partial u_l}=\sum_{l=1}^{N}\lambda_k^{(l)}(e^{ju_l}-1)h_1(k,\mathbf{u}). \tag{5}$$

The general solution to the system of equations (5) has the form:

$$h_1(k,\mathbf{u})=C(k)\exp\left\{\sum_{l=1}^{N}\frac{\lambda_k^{(l)}}{\mu_l}(e^{ju_l}-1)\right\},k=1,\dots,K. \tag{6}$$

The constant $C(k)$ is determined by the following initial conditions:

$$h_1(k,\mathbf{0})=C(k)=r(k),\quad k=1,\dots,K.$$

Finally, we get

$$h(k,\mathbf{u},\varepsilon)\approx h_1(k,\mathbf{u})=r(k)\exp\left\{\sum_{l=1}^{N}\frac{\lambda_k^{(l)}}{\mu_l}(e^{ju_l}-1)\right\}. \tag{7}$$

Summing (7) over all $k=1,\dots,K$, we find that the characteristic function of the joint probability distribution of the number of occupied service block of each type has the following form:

$$h(\mathbf{u})\approx\sum_{k=0}^{N}r(k)\exp\left\{\sum_{l=1}^{N}\frac{\lambda_k^{(l)}}{\mu_l}(e^{ju_l}-1)\right\}. \tag{8}$$

The one-dimensional characteristic functions of the stationary probability distribution of the number of occupied service blocks of each type for $s=1,...,N$ have the following form:

$$h(u_s)=M\{e^{ju_si_s}\}\approx\sum_{k=0}^{N}r(k)\exp\left\{\frac{\lambda_k^{(s)}}{\mu_s}(e^{ju_s}-1)\right\}= \\ =\sum_{i=0}^{\infty}(e^{ju_s})^i\sum_{k=1}^{K}r(k)\frac{\left(\frac{\lambda_k^{(s)}}{\mu_s}\right)^i}{i!}e^{-\frac{\lambda_k^{(s)}}{\mu_s}}. \tag{9}$$

As a result, the weighted sum of Poisson distributions represents the marginal probability distributions of the number of occupied blocks of each type:

$$\lim_{\varepsilon \to 0} \Pi(i_s, \varepsilon) = \Pi(i_s) = \sum_{k=1}^{K} r(k) \frac{(\rho_k^{(s)})^{i_s}}{i_s!} e^{-\rho_k^{(s)}}, \quad \text{where} \;\; \rho_k^{(s)} = \lambda_k^{(s)}/\mu_s. \tag{10}$$

Likewise, it can be demonstrated that the probability distribution in two dimensions is provided by

$$\Pi(i_s, i_l) = \sum_{k=1}^{K} r(k) \frac{(\rho_k^{(s)})^{i_s}}{i_s!} \frac{(\rho_k^{(l)})^{i_l}}{i_l!} e^{-\rho_k^{(s)}} e^{-\rho_k^{(l)}}, l, s = 1, \ldots, N. \tag{11}$$

4 Conclusion

This paper presents a mathematical model of a multi-thread data transmission system in the form of a heterogeneous infinite-server queuing system with several incoming flows of different modalities.

Although network resources are typically constrained in practice, there is a strong presumption that there is an infinite number of channels, since in this case there are no losses in the system. The obtained asymptotic probability distribution and expressions for moments make it possible to estimate the optimally required amount of resources of each block for a system with a limited resource, providing a given probability of losses.

It is planned to implement a simulation model to compare the asymptotic and empirical probability distributions of the studied processes, as well as to solve optimization issues.

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