Evsey V. Morozov
Alexander S. Rumyantsev
Oleg V. Lukashenko (Eds.)

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Anatolii Puhalskii

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On a Metastable Loss Network

Anatolii Puhalskii

IITP RAS,
Moscow, Russia,
puhalski@iitp.ru

Abstract. I will discuss large deviation properties of a mean field thermodynamic limit of a multiclass loss network. The network may have several stable equilibria. The invariant measure of the network process is shown to obey a large deviation principle. The network is metastable in that it spends exponentially long periods of time in the neighbourhoods of stable equilibria. As a case study, a two-class network with two stable and one unstable equilibria is examined.

1 Introduction

The following model of a cellular network was studied in Antunes et al. [1]. There are n nodes of capacity C each. Customers of K classes arrive at the nodes according to Poisson processes of respective rates α_k , $1 \leq k \leq K$. On arrival at a node, a class k customer occupies A_k units of the node's capacity, being rejected and removed from the network if the required capacity is not available. On acceptance, the customer stays at the node for an exponentially distributed length of time with mean $1/\gamma_k$ and then moves to another node, a destination being chosen uniformly at random. As on arrival, rejection occurs within the network when less than A_k units of unused capacity are available at a destination node. A class k customer may also leave the network after an exponentially distributed length of time of mean $1/\delta_k$. The arrival processes, sojourn times at the nodes, sojourn times in the network and routing decisions are independent.

Antunes et al. [1] obtained a law of large numbers for the process of the proportions of nodes with a given population, as the number of nodes goes to infinity. They analysed stability properties of the limit dynamical system and showed that it may have several stable equilibria. In Tibi [4], building on Freidlin and Wentzell [2], it was argued that the network process would spend exponentially long periods of time in the neighbourhoods of stable equilibria implying that the network is metastable. Unfortunately, the analysis in Tibi [4] is not complete.

This model stands out because mean field behaviour arises in the limit only, the interactions within the network being local. Usually, when mean–field models are considered, the mean-field interaction is built in the hypotheses. In a similar vein, in the available literature multistability of a dynamical system, resulting in metastability, is assumed extraneously, for the most part, whereas in this model it is an intrinsic feature, too.

The state of node i at time t is described by the vector

$$X_i^{(n)}(t) = \left(X_{i,1}^{(n)}(t), \ldots, X_{i,K}^{(n)}(t)\right),$$

whose k-th entry records the number of class k customers at the node. The process $X_i^{(n)} = (X_i^{(n)}(t)\,, t \geq 0)$ takes values in the set

$$\Theta = \left\{ \theta = (\theta_1, \ldots, \theta_K) \in \mathbb{Z}_+^K : \sum_{k=1}^K \theta_k A_k \leq C \right\}.$$

It is assumed that $|\Theta| \geq 2$, $|\Theta|$ denoting the cardinality of Θ. Let $Y_\theta^{(n)}(t)$ represent the proportion of nodes with θ as the population vector, i.e.,

$$Y_\theta^{(n)}(t) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\left\{X_i^{(n)}(t) = \theta\right\}}$$

and let $Y^{(n)}(t) = (Y_\theta^{(n)}(t)\,, \theta \in \Theta)$, where $\mathbf{1}_\Xi$ denotes the indicator of event Ξ. As

$$\sum_{\theta \in \Theta} Y_\theta^{(n)}(t) = 1\,,$$

the process $Y^{(n)} = (Y^{(n)}(t)\,, t \geq 0)$ is a Markov process with values in the discrete simplex $\mathbb{S}_{|\Theta|}^{(n)} = \{y = (y_\theta,\, \theta \in \Theta) : \sum_{\theta \in \Theta} y_\theta = 1\,, y_\theta \geq 0\,, n y_\theta \text{ is an integer}\}$. Let $\mathbb{S}_{|\Theta|} = \{y = (y_\theta\,, \theta \in \Theta) : \sum_{\theta \in \Theta} y_\theta = 1\,, y_\theta \geq 0\}$. It follows from the results in Antunes et al. [1] that if the sequence $Y^{(n)}(0)$ converges in probability to $\hat{y} \in \mathbb{S}_{|\Theta|}$, as $n \to \infty$, then the sequence $Y^{(n)}$ converges in probability uniformly over compact intervals to the solution $\mathbf{y} = (\mathbf{y}(t)\,, t \geq 0)$ of the initial value problem

$$\dot{\mathbf{y}}_\theta(t) = V_\theta(\mathbf{y}(t)) \tag{1}$$

and $\mathbf{y}(0) = \hat{y}$, where $\mathbf{y}(t) = (\mathbf{y}_\theta(t)\,, \theta \in \Theta) \in \mathbb{S}_{|\Theta|}$ and, for $y = (y_\theta\,, \theta \in \Theta) \in \mathbb{S}_{|\Theta|}$,

$$\begin{aligned} V_\theta(y) = \sum_{k=1}^K \Big((\alpha_k + \sum_{\theta' \in \Theta} \theta'_k \gamma_k y_{\theta'}) y_{\theta - e_k} + (\delta_k + \gamma_k)(\theta_k + 1) y_{\theta + e_k} \\ - (\alpha_k + (\delta_k + \gamma_k)\theta_k + \sum_{\theta' \in \Theta} \theta'_k \gamma_k y_{\theta'}) y_\theta \Big)\,, \end{aligned}$$

with an overdot denoting a time derivative, e_k denoting the kth vector of the canonical basis of $\mathbb{R}^K$ and with the convention that $y_{\theta \pm e_k} = 0$ if $\theta \pm e_k \notin \Theta$.

Both $Y^{(n)}(t)$ and $\mathbf{y}(t)$ are probability distributions on Θ. The equilibrium points of (1) are given by an Erlang formula for the stationary distribution of an $M/M/C/C$ queue. More specifically, for $\rho = (\rho_1, \dots, \rho_K) \in \mathbb{R}_+^K$, let a probability distribution $\nu(\rho) = (\nu_\theta(\rho)\,, \theta \in \Theta)$ on Θ be defined as

$$\nu_\theta(\rho) = \frac{1}{Z(\rho)} \prod_{k=1}^{K} \frac{\rho_k^{\theta_k}}{\theta_k!}\,, \tag{2}$$

$Z(\rho)$ being a normalising constant. If, for $k = 1, 2, \dots, K$,

$$\rho_k = \frac{\alpha_k + \gamma_k \sum_{\theta\in\Theta} \theta_k \nu_\theta(\rho)}{\gamma_k + \delta_k}\,, \tag{3}$$

then $y = \nu(\rho)$ is an equilibrium of (1). Every equilibrium is of this form. The existence of solutions to (2) and (3) is proved via an application of Brouwer's fixed point theorem. On the other hand, uniqueness of an equilibrium for (1) might not hold and in Antunes et al. [1] an example of a network with no less than two stable equilibria is provided, so, metastability is likely to occur. Substantiation of this intuition requires a trajectorial large deviation principle (LDP) for the sequence of $Y^{(n)}$ as random elements of the Skorohod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^\Theta)$.

2 A trajectorial LDP

Let, for $y = (y_\theta\,, \theta \in \Theta) \in \mathbb{R}^\Theta$, $z = (z_\theta\,, \theta \in \Theta) \in \mathbb{R}^\Theta$, and $\lambda = (\lambda_\theta\,, \theta \in \Theta) \in \mathbb{R}^\Theta$,

$$\begin{aligned} H(y,\lambda) = &\sum_{k=1}^{K} \sum_{\theta\in\Theta_k^+} (e^{\lambda_{\theta+e_k}-\lambda_\theta} - 1)\alpha_k y_\theta \\ &+ \sum_{k=1}^{K} \sum_{\theta\in\Theta_k^-} (e^{\lambda_{\theta-e_k}-\lambda_\theta} - 1)(\delta_k + \gamma_k \sum_{\theta'\in\Theta\setminus\Theta_k^+} y_{\theta'})\theta_k y_\theta \\ &+ \sum_{k=1}^{K} \sum_{\theta'\in\Theta_k^+,\,\theta\in\Theta_k^-} (e^{\lambda_{\theta'+e_k}-\lambda_{\theta'}+\lambda_{\theta-e_k}-\lambda_\theta} - 1)\theta_k\gamma_k y_\theta y_{\theta'} \end{aligned}$$

and

$$L(y,z) = \sup_{\lambda\in\mathbb{R}^\Theta} \big(\lambda\cdot z - H(y,\lambda)\big)\,,$$

where $\Theta_k^\pm = \{\theta \in \Theta : \theta \pm e_k \in \Theta\}$ and " $\cdot$ " is used to denote an inner product. Let $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^\Theta)$ represent the Skorohod space of right continuous $\mathbb{R}^\Theta$–valued functions on $\mathbb{R}_+$ with lefthand limits and let $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^\Theta)$ represent the space of continuous $\mathbb{R}^\Theta$–valued functions on $\mathbb{R}_+$, respectively.

Theorem 1. *Let $y^{(n)} \in \mathbb{S}_{|\Theta|}^{(n)}$, $y \in \mathbb{S}_{|\Theta|}$, and $y^{(n)} \to y$ as $n \to \infty$. Then the sequence $Y^{(n)}$ with $Y^{(n)}(0) = y^{(n)}$ obeys an LDP in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^\Theta)$ with deviation function*

$$I_y(\mathbf{y}) = \int_0^\infty L(\mathbf{y}(s), \dot{\mathbf{y}}(s))\, ds\,, \tag{4}$$

provided $\mathbf{y} = (\mathbf{y}(t)\,, t \geq 0)$ *is an absolutely continuous function taking values in* $\mathbb{S}_{|\Theta|}$ *with* $\mathbf{y}(0) = y$ *, and* $I_y(\mathbf{y}) = \infty$ *, otherwise.*

More explicitly, the theorem asserts that the sets $\{\mathbf{y} \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^\Theta) : I_y(\mathbf{y}) \leq \beta\}$ are compact for all $\beta \geq 0$ and that, for any Borel set $W \subset \mathbb{D}(\mathbb{R}_+, \mathbb{R}^\Theta)$ such that $\inf_{\mathbf{y} \in \mathrm{int}\, W} I_y(\mathbf{y}) = \inf_{\mathbf{y} \in \mathrm{cl}\, W} I_y(\mathbf{y})$, $(1/n) \ln \mathbf{P}(Y^{(n)} \in W) \to -\inf_{\mathbf{y} \in W} I_y(\mathbf{y})$, as $n \to \infty$, where int and cl denote the interior and closure of a set, respectively.

3 Large deviations of the invariant measure. Metastability

Being irreducible and having a finite state space, the process $Y^{(n)}$ possesses a unique invariant measure on $\mathbb{S}^{(n)}_{|\Theta|}$, which is denoted by $\mu^{(n)}$. It is convenient to extend $\mu^{(n)}$ to the whole of $\mathbb{S}_{|\Theta|}$ by letting $\mu^{(n)}(\mathbb{S}_{|\Theta|} \setminus \mathbb{S}^{(n)}_{|\Theta|}) = 0$. Let, for $t > 0$, $y \in \mathbb{S}_{|\Theta|}$ and $y' \in \mathbb{S}_{|\Theta|}$,

$$\Phi_t(y, y') = \inf_{\substack{\mathbf{y} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^\Theta): \\ \mathbf{y}(0) = y,\, \mathbf{y}(t) = y'}} I_y(\mathbf{y})\,.$$

Given $\nu(\rho) = (\nu_\theta(\rho)\,, \theta \in \Theta)$, as defined in (2), and $y \in \mathbb{S}_{|\Theta|}$, let

$$\Phi(\nu(\rho), y) = \lim_{t \to \infty} \Phi_t(\nu(\rho), y) = \inf_{\substack{\mathbf{y} \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^\Theta): \\ \mathbf{y}(0) = \nu(\rho),\, \mathbf{y}(t) = y \text{ for some } t}} I_{\nu(\rho)}(\mathbf{y})\,.$$

Let A denote the set of solutions ρ of (2) and (3). For $\rho \in A$, let $G(\rho)$ denote the set of directed graphs that are in-trees with root ρ on the vertex set A. Thus, for every $\rho' \in A$ and $q \in G(\rho)$, there is a unique directed path from ρ' to ρ in q. For $q \in G(\rho)$, let $E(q)$ denote the set of edges of q. Define

$$J(\nu(\rho)) = \inf_{q \in G(\rho)} \sum_{(\rho', \rho'') \in E(q)} \Phi(\nu(\rho'), \nu(\rho'')) - \inf_{\tilde{\rho} \in A} \inf_{q \in G(\tilde{\rho})} \sum_{(\rho', \rho'') \in E(q)} \Phi(\nu(\rho'), \nu(\rho''))\,. \tag{5}$$

Theorem 2. *Suppose that the equations in* (2) *and* (3) *admit finitely many solutions* $\rho = (\rho_1, \ldots, \rho_K)$ *. Then, the measures* $\mu^{(n)}$ *satisfy an LDP in* $\mathbb{S}_{|\Theta|}$ *for the topology of weak convergence with a continuous deviation function*

$$J(y) = \inf_{\rho \in A} \big(J(\nu(\rho)) + \Phi(\nu(\rho), y)\big)\,. \tag{6}$$

The next result concerns metastability. It is in the spirit of Freidlin and Wentzell [2], see also Shwartz and Weiss [3]. Let P_y and E_y denote probability and expectation, respectively, that correspond to the initial condition $Y^{(n)}(0) = y$.

Theorem 3. *Let* $\nu(\rho)$ *be an equilibrium of* (1) *and let* D *be an open subset of* $\mathbb{S}_{|\Theta|}$ *that contains* $\nu(\rho)$ *. Suppose that the solutions of* (1) *with initial conditions* *in some neighbourhood of D converge to $\nu(\rho)$ and stay in D when started in D. Let $\tau^{(n)} = \inf\{t \geq 0 : Y^{(n)}(t) \notin D\}$. Let $y^{(n)} \in D \cap S^{(n)}_{|\Theta|}$. If $y^{(n)} \to y \in D$, as $n \to \infty$, then*

$$P_{y^{(n)}}\big(|\frac{1}{n}\ln \tau^{(n)} - U| > \kappa\big) \to 0$$

and

$$\frac{1}{n}\ln E_{y^{(n)}}(\tau^{(n)})^m \to mU\,,$$

where $m \in \mathbb{N}$,

$$U = \inf_{y' \notin D} \Phi(\nu(\rho), y')$$

and $\kappa > 0$ is otherwise arbitrary.

In Antunes et al. [1] stability of equilibria is tackled via the Lyapunov function

$$g(y) = \sum_{\theta \in \Theta} y_\theta \ln(\prod_{k=1}^K \theta_k!\, y_\theta) - \sum_{k=1}^K \frac{\delta_k + \gamma_k}{\gamma_k}\,(u \ln u - u)\Big|_{u=\alpha_k/(\delta_k+\gamma_k)}^{u=(\alpha_k+\gamma_k\sum_{\theta\in\Theta}\theta_k y_\theta)/(\delta_k+\gamma_k)}\,.$$

In the interior of $\mathbb{S}_{|\Theta|}$, see Antunes et al. [1],

$$\nabla g(y)\cdot V(y) = \sum_{k=1}^K \sum_{\theta\in\Theta}((\delta_k+\gamma_k)\theta_k y_\theta - b_k(y)) \ln \frac{b_k(y)}{(\delta_k+\gamma_k)\theta_k y_\theta}\,,$$

where

$$b_k(y) = \alpha_k + \gamma_k \sum_{\theta\in\Theta} \theta_k y_\theta\,.$$

Hence, $\nabla g(y)\cdot V(y) \leq 0$ so that $g(\mathbf{y}(t))$ is nonincreasing with t along solutions of (1) and $\nabla g(y)\cdot V(y) < 0$ provided y is not an equilibrium of (1). If y is a local minimum of g, it is an asymptotically stable equilibrium. In order "to reduce dimension", Antunes et al. [1] introduce the function

$$\phi(\rho) = -\ln Z(\rho) + \sum_{k=1}^K\big(\frac{\gamma_k+\delta_k}{\gamma_k}\,\rho_k - \frac{\alpha_k}{\gamma_k}\,\ln \rho_k\big)\,.$$

By Theorem 3 in Antunes et al. [1], $\rho \in \mathbb{R}_+^K$ is a local minimum of ϕ if and only if $\nu(\rho)$ is a local minimum of g; if ρ is a saddle point of ϕ, then $\nu(\rho)$ is a saddle point of g. Besides, if $\nu(\rho)$ is an equilibrium, then

$$g(\nu(\rho)) = \phi(\rho) + \sum_{k=1}^K \frac{\alpha_k}{\gamma_k}\big(\ln\frac{\alpha_k}{\gamma_k+\delta_k} - 1\big)\,.$$

An example of bistability along the lines of the one in Antunes et al. [1] is analysed next. Let $K = 2$. Antunes et al. [1] show that, for a certain choice of parameters there are at least two stable equilibria. Suppose that class 1 customers require one unit of capacity, so, $A_1 = 1$ whereas class 2 customers require the whole capacity, so, $A_2 = C$. Accordingly, class 1 and class 2 customers cannot coexist at the same node. It stands to reason that there could be two stable states where class 1 customers are prevalent or class 2 customers are prevalent, respectively. This is substantiated next.

By hypotheses, θ_1 takes values in the set $\{0, 1, \ldots, C\}$ and θ_2 takes values in $\{0, 1\}$. Then, with $\rho = (\rho_1, \rho_2)$,

$$Z(\rho) = \sum_{i=0}^{C} \frac{\rho_1^i}{i!} + \rho_2 \tag{7}$$

and

$$\phi(\rho) = -\ln\Bigl(\sum_{i=0}^{C} \frac{\rho_1^i}{i!} + \rho_2\Bigr) + \frac{\gamma_1 + \delta_1}{\gamma_1}\,\rho_1 + \frac{\gamma_2 + \delta_2}{\gamma_2}\,\rho_2 - \frac{\alpha_1}{\gamma_1}\,\ln\rho_1 - \frac{\alpha_2}{\gamma_2}\,\ln\rho_2\,.$$

By (2) and (3) with $k = 1$,

$$Z(\rho) = \frac{\gamma_1\rho_1(\rho_2 + \rho_1^C/C!)}{\alpha_1 - \delta_1\rho_1} \tag{8}$$

(as $\sum_{\theta\in\Theta}\theta_1\nu_\theta(\rho) < \rho_1$, $\alpha_1 > \delta_1\rho_1$). By (2) and (3) with $k = 2$, ρ_2 satisfies the quadratic equation

$$\gamma_1(\gamma_2+\delta_2)\rho_2^2 + \Bigl(\gamma_1(\gamma_2+\delta_2)\frac{\rho_1^C}{C!} - \alpha_2\gamma_1 - \gamma_2\bigl(\frac{\alpha_1}{\rho_1} - \delta_1\bigr)\Bigr)\rho_2 - \alpha_2\gamma_1\,\frac{\rho_1^C}{C!} = 0\,.$$

Hence,

$$\rho_2(\rho_1) = \frac{1}{2\gamma_1(\gamma_2+\delta_2)}\Bigl(-\gamma_1(\gamma_2+\delta_2)\frac{\rho_1^C}{C!} + \alpha_2\gamma_1 + \gamma_2\bigl(\frac{\alpha_1}{\rho_1} - \delta_1\bigr) + \sqrt{\Bigl(\gamma_1(\gamma_2+\delta_2)\frac{\rho_1^C}{C!} - \alpha_2\gamma_1 - \gamma_2\bigl(\frac{\alpha_1}{\rho_1} - \delta_1\bigr)\Bigr)^2 + 4\gamma_1^2(\gamma_2+\delta_2)\alpha_2\,\frac{\rho_1^C}{C!}}\;\Bigr)\,. \tag{9}$$

Equating the righthand sides of (7) and (8) yields

$$h(\rho) = 0\,,$$

where

$$h(\rho) = \sum_{i=0}^{C}\frac{\rho_1^i}{i!} + \rho_2 - \frac{\gamma_1\rho_1}{\alpha_1 - \delta_1\rho_1}\Bigl(\frac{\rho_1^C}{C!} + \rho_2\Bigr)\,. \tag{10}$$

Let $C = 20$, $\alpha_1 = .5$, $\alpha_2 = 9$, $\gamma_1 = \gamma_2 = 1$ and $\delta_1 = \delta_2 = .01$. (The parameters for which Antunes et al. [1] show the existence of two stable equilibria are $C = 20$, $\alpha_1 = .68$, $\alpha_2 = 9$, $\gamma_1 = \gamma_2 = 1$ and $\delta_1 = \delta_2 = 0$. It is of interest to allow nonzero δ_1 and (or) δ_2.) For ρ_1 close to zero, the term α_1/ρ_1 on the right of (9) dominates, ρ_2 decreases rapidly as ρ_1 increases, so does the righthand side of (10), its first zero being $\rho_1 \approx .5966$. The righthand side of (10) keeps decreasing as the leftmost sum starts taking over until it reaches a minimum of approximately -23.556 at $\rho_1 \approx 2.8861$ and begins to increase and crosses the zero level for a second time for $\rho_1 \approx 4.1786$. It keeps growing reaching a maximum of approximately $1.5794 \cdot 10^5$ at $\rho_1 \approx 12.896$ until another change of the dominating term when $-\gamma_1\rho_1/(\alpha_1 - \rho_1\delta_1)\,\rho_1^C/C!$ takes over. The righthand side of (10) then plunges to $-\infty$, crossing the zero level for a third time at $\rho_1 \approx 13.72715$ in the process. Graphs in Fig.1 and Fig.2 provide an illustration. Thus, all in all, there are three equilibria: $\rho^{(1)} \approx (.5966, 8.8293)$,

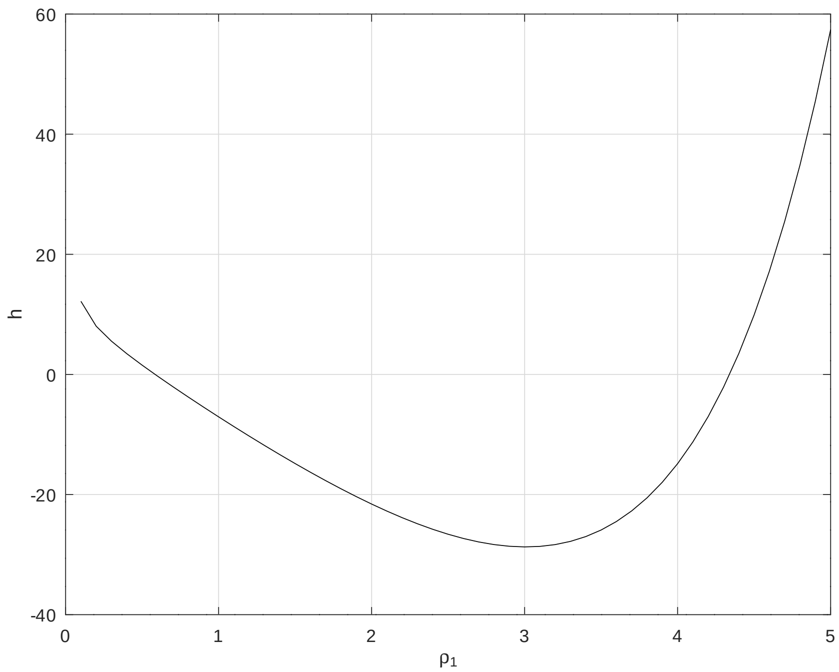

Fig. 1. Function $h(\rho_1, \rho_2(\rho_1))$ for ρ_1 small

$\rho^{(2)} \approx (4.1786, 8.1115)$, and $\rho^{(3)} \approx (13.72715, 8.9906)$. The second derivatives of ϕ are $\nabla^2\phi(\rho^{(1)}) \approx \begin{pmatrix} 1.2633 & 0.016025 \\ 0.016025 & 0.1243 \end{pmatrix}$, $\nabla^2\phi(\rho^{(3)}) \approx \begin{pmatrix} 0.015412 & 1.1085 \cdot 10^{-6} \\ 1.1085 \cdot 10^{-6} & 0.1113 \end{pmatrix}$, and $\nabla^2\phi(\rho^{(2)}) \approx \begin{pmatrix} -0.069679 & 0.0121200 \\ 0.012120 & 0.1370 \end{pmatrix}$, the eigenvalues in the latter case being -0.070387 and 0.137708 approximately. With $\nabla^2\phi(\rho^{(1)})$ and $\nabla^2\phi(\rho^{(3)})$ being positive definite, $\rho^{(1)}$ and $\rho^{(3)}$ are local minima, whereas $\rho^{(2)}$ is a saddle point. A 3D mesh plot of $\phi(\rho)$ with a contour plot underneath is in Fig.3. Therefore, $\nu(\rho^{(1)}))$ and $\nu(\rho^{(3)})$ are asymptotically stable equilibria so that the network pro-

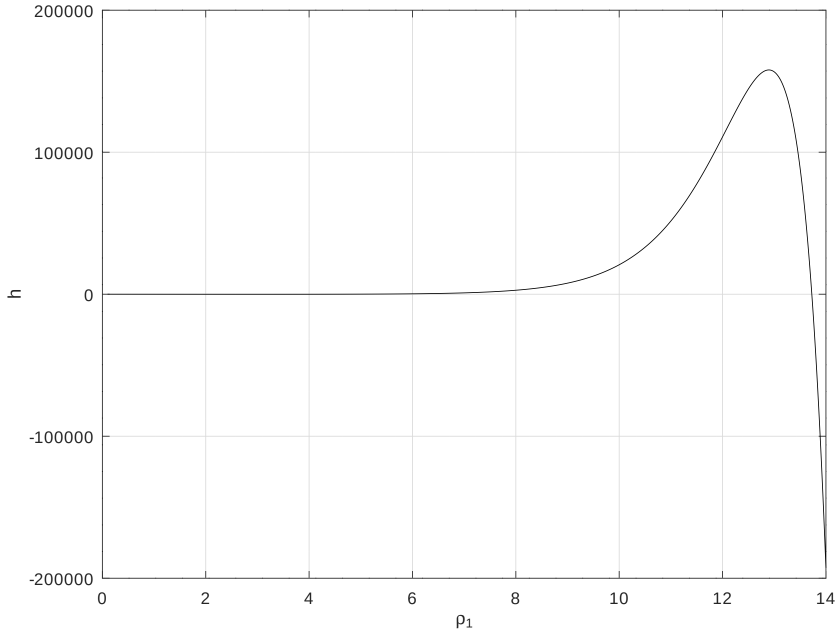

Fig. 2. Function $h(\rho_1, \rho_2(\rho_1))$ globally

cess spends exponentially long periods of time in the neighbourhoods of those equilibria, while $\nu(\rho^{(2)})$ is an unstable equilibrium.

The expected number of class k customers being

$$EQ_k = \rho_k(1 - \sum_{\theta:\, \sum_{k'} A_{k'}\theta_{k'} > C - A_k} \nu_\theta(\rho))$$

implies that the average numbers of class 1 and class 2 customers are

$$EQ_1 = \rho_1(1 - \frac{1}{Z(\rho)}\frac{\rho_1^C}{C!})$$

and

$$EQ_2 = \rho_2(1 - \frac{1}{Z(\rho)}(\sum_{\theta_1=1}^{C}\frac{\rho_1^{\theta_1}}{\theta_1!} + \rho_2)) = \frac{\rho_2}{Z(\rho)}\,,$$

respectively.

For the stable equilibria, calculations yield $(EQ_1^{(1)}, EQ_2^{(1)}) \approx (.5966, .8281)$ and $(EQ_1^{(3)}, EQ_2^{(3)}) \approx (13.365, 1.0235 \cdot 10^{-5})$. Thus, for $\rho = \rho^{(1)}$, class 2 customers are prevalent and, for $\rho = \rho^{(3)}$, class 1 customers are prevalent. (The pattern of $h(\rho)$ first decreasing, then increasing and decreasing again is sensitive to the values of δ_1 and δ_2. When $\delta_1 = \delta_2 = .1$, only a downward trend is present, so, there is only one equilibrium.)

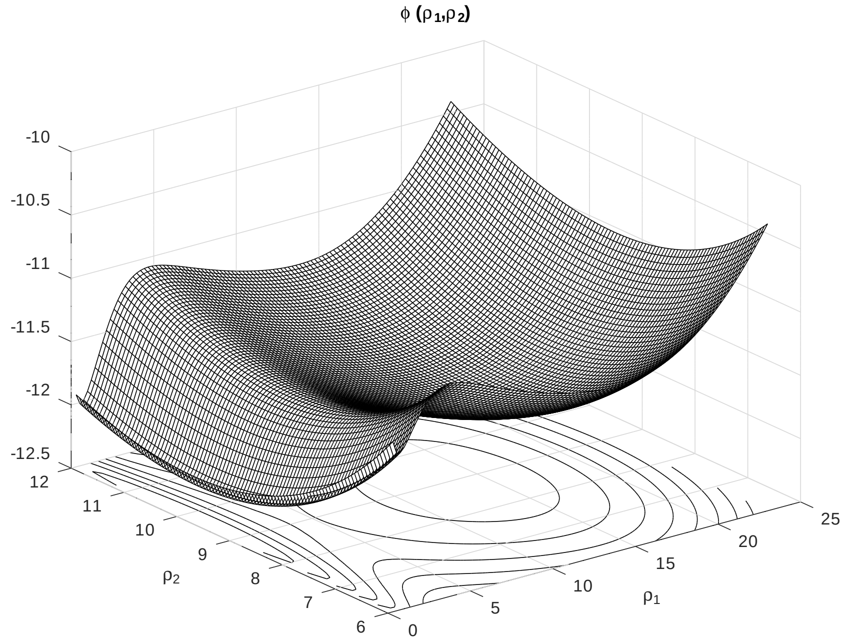

Fig. 3. 3D mesh plot and contour plot of $\phi(\rho_1, \rho_2)$

Also, calculations yield $\phi(\rho^{(1)})\!=\!-12.284$, $\phi(\rho^{(2)})\!=\!-11.560$, $\phi(\rho^{(3)})\!=\!-12.043$ so that $g(\nu(\rho^{(2)})) > g(\nu(\rho^{(1)})) \vee g(\nu(\rho^{(3)}))$. Thus, if $\mathbf{y}(0) = \nu(\rho^{(2)})$ experiences a small displacement Δ at time zero in a direction collinear with the direction of the eigenvector with a negative eigenvalue, then $g(y+\Delta) < g(y)$, so that the associated trajectory $\mathbf{y}(t)$ will end up in one of the equilibria $\nu(\rho^{(1)})$ or $\nu(\rho^{(3)})$. Hence, either $\Phi(\nu(\rho^{(2)}), \nu(\rho^{(1)}) = 0$ or $\Phi(\nu(\rho^{(2)}), \nu(\rho^{(3)})) = 0$.

On denoting $\nu^{(i)} = \nu(\rho^{(i)})$, by (5),

$$
\begin{aligned}
J(\nu^{(1)}) =&(\Phi(\nu^{(2)}, \nu^{(1)}) + \Phi(\nu^{(3)}, \nu^{(2)})) \wedge (\Phi(\nu^{(2)}, \nu^{(1)}) + \Phi(\nu^{(3)}, \nu^{(1)}))\\
&\wedge(\Phi(\nu^{(3)}, \nu^{(1)}) + \Phi(\nu^{(2)}, \nu^{(3)})) - \Psi\,,\\
J(\nu^{(2)}) =&(\Phi(\nu^{(1)}, \nu^{(2)}) + \Phi(\nu^{(3)}, \nu^{(1)})) \wedge (\Phi(\nu^{(1)}, \nu^{(2)}) + \Phi(\nu^{(3)}, \nu^{(2)}))\\
&\wedge(\Phi(\nu^{(3)}, \nu^{(2)}) + \Phi(\nu^{(1)}, \nu^{(3)})) - \Psi\,,\\
J(\nu^{(3)}) =&(\Phi(\nu^{(2)}, \nu^{(3)}) + \Phi(\nu^{(1)}, \nu^{(2)})) \wedge (\Phi(\nu^{(2)}, \nu^{(3)}) + \Phi(\nu^{(1)}, \nu^{(3)}))\\
&\wedge(\Phi(\nu^{(1)}, \nu^{(3)}) + \Phi(\nu^{(2)}, \nu^{(1)})) - \Psi\,,
\end{aligned}
$$

where

$$\begin{aligned}\Psi =&(\Phi(\nu^{(2)},\nu^{(1)})+\Phi(\nu^{(3)},\nu^{(2)}))\wedge(\Phi(\nu^{(2)},\nu^{(1)})+\Phi(\nu^{(3)},\nu^{(1)}))\\ &\wedge(\Phi(\nu^{(3)},\nu^{(1)})+\Phi(\nu^{(2)},\nu^{(3)}))\wedge(\Phi(\nu^{(1)},\nu^{(2)})+\Phi(\nu^{(3)},\nu^{(1)}))\\ &\wedge(\Phi(\nu^{(1)},\nu^{(2)})+\Phi(\nu^{(3)},\nu^{(2)}))\wedge(\Phi(\nu^{(3)},\nu^{(2)})+\Phi(\nu^{(1)},\nu^{(3)}))\\ &\wedge(\Phi(\nu^{(2)},\nu^{(3)})+\Phi(\nu^{(1)},\nu^{(2)}))\wedge(\Phi(\nu^{(2)},\nu^{(3)})+\Phi(\nu^{(1)},\nu^{(3)}))\\ &\wedge(\Phi(\nu^{(1)},\nu^{(3)})+\Phi(\nu^{(2)},\nu^{(1)}))\,.\end{aligned}$$

As a consequence, if $\Phi(\nu^{(2)},\nu^{(1)})=0$, then

$$\begin{aligned}J(\nu^{(1)}) =&\Phi(\nu^{(3)},\nu^{(2)})\wedge\Phi(\nu^{(3)},\nu^{(1)})-\Psi\,,\\ J(\nu^{(2)}) =&(\Phi(\nu^{(1)},\nu^{(2)})+\Phi(\nu^{(3)},\nu^{(1)}))\wedge(\Phi(\nu^{(1)},\nu^{(2)})+\Phi(\nu^{(3)},\nu^{(2)}))\\ &\wedge(\Phi(\nu^{(3)},\nu^{(2)})+\Phi(\nu^{(1)},\nu^{(3)}))-\Psi\,,\\ J(\nu^{(3)}) =&(\Phi(\nu^{(2)},\nu^{(3)})+\Phi(\nu^{(1)},\nu^{(2)}))\wedge\Phi(\nu^{(1)},\nu^{(3)})-\Psi\end{aligned}$$

and

$$\Psi=\Phi(\nu^{(3)},\nu^{(2)})\wedge\Phi(\nu^{(3)},\nu^{(1)})\wedge(\Phi(\nu^{(2)},\nu^{(3)})+\Phi(\nu^{(1)},\nu^{(2)}))\wedge\Phi(\nu^{(1)},\nu^{(3)})\,.$$

It is being conjectured that, furthermore, $\Phi(\nu^{(2)},\nu^{(1)})=\Phi(\nu^{(2)},\nu^{(3)})=0$ and that $\Phi(\nu^{(3)},\nu^{(2)})\le\Phi(\nu^{(3)},\nu^{(1)})$ and $\Phi(\nu^{(1)},\nu^{(2)})\le\Phi(\nu^{(1)},\nu^{(3)})$, in which case the expressions above simplify.

4 Conclusion

The results of this contribution provide a basis for quantifying effects caused by the metastability of communication networks.

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