On Rate Matrix R of $G/M/1$-type Markov Process

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Abstract. In this paper we establish an upper bound for absolute value of the determinant of rate matrix R used in matrix-geometric solution for steady-state probabilities of a structured $G/M/1$-type Markov process. This result, although being mostly theoretical, may be useful to establish bounds for the geometrical decay of steady-state probabilities useful in many practical problems.

1 Introduction

In this short paper we address a particular problem related to matrix-analytic method (MAM) of stochastic simulation. The MAM is widely used to study Markov processes of special structure, with many successful applications to stochastic models of queueing systems [11,8]. Practical applications of MAM cover many areas in modern computing and communication systems, such as Internet of Things [7], high-performance [13] and distributed [4] computing systems.

The key component of the MAM analysis of a $G/M/1$-type system in stationary regime is obtaining a minimal nonnegative matrix R solving the following matrix series equation

$$\sum_{i=0}^{\infty} R^i A^{(i)} = \mathbf{0},$$

or (if motivated by the model) the following matrix polynomial equation of power $N \geqslant 2$ which we focus our attention on:

$$P(R) := \sum_{i=0}^{N} R^i A^{(i)} = \mathbf{0}. \tag{1}$$

In general, the solution R may be obtained either numerically [2], or by invariant subspaces approach [10], and finally, by spectral decomposition [6]. However,

apart from componentwise non-negativity, not many properties of the key matrix R are known in general [5].

In this paper we obtain an upper bound for the absolute value of determinant of R in terms of determinants of the submatrices of infinitesimal generator matrix of Markov process, considered to be known in advance from the model properties. The corresponding theorem is proven in Section 2. In Section 3 we discuss the applicability of this theoretical result to practical problems, and give some conclusions.

2 Upper bound on the determinant of key matrix R

The system studied by MAM is usually modelled as a two-dimensional continuous time Markov chain $\{(X(t), Y(t)), t \geqslant 0\}$ with countable state space $E := \{(0, j), j = 1, \ldots, m_0\} \cup \{(i, j), i \geqslant 1, j = 1, \ldots, m\}$, where the so-called *phase* $Y(t)$ may take one of m (or m_0 for boundary states) values and *level* $X(t)$ may be increased/decreased at each transition. The state space E can be partitioned into *levels* with level $n \geqslant 1$ being the subset $\{(n, j), j = 1, \ldots, m\} \subset E$. In many fields of interest, it is assumed that the level is increased by at most one, and decreased by at most $N-1$ (we focus on the case $N < \infty$) units at each transition epoch. These models belong to the so-called structured $G/M/1$-type Markov processes, extensively studied in [11], with the natural example of such a process being the queue length process of an $G/M/1$ queue, embedded at arrival epochs. The infinitesimal generator matrix of a structured $G/M/1$-type process has the following block-multidiagonal representation

$$Q = \begin{pmatrix} A^{0,0} & A^{0,1} & 0 & 0 & \ldots \\ A^{1,0} & A^{1,1} & A^{(0)} & 0 & \ldots \\ A^{2,0} & A^{(2)} & A^{(1)} & A^{(0)} & \ldots \\ A^{3,0} & A^{(3)} & A^{(2)} & A^{(1)} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{0} & A^{(N)} & A^{(N-1)} & A^{(N-2)} & \ldots \\ \mathbf{0} & \mathbf{0} & A^{(N)} & A^{(N-1)} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2}$$

where $A^{(i)}, i = 0, \ldots, N$ are square matrices of order m, satisfying the balance equation

$$A\mathbf{1} = \mathbf{0}, \qquad \text{where } A := \sum_{i=0}^{N} A^{(i)}, \tag{3}$$

$\mathbf{1}$ ($\mathbf{0}$) is the vector of ones (zeroes) of corresponding dimension, $A^{0,0}$ is a square matrix of order m_0 and $A^{i,0}, A^{0,1}$ are possibly rectangular matrices. (Recall that for these type of processes the off-diagonal elements of matrix Q, i.e. the rates of transitions of the chain, are nonnegative.)

The key component of the method is to obtain the steady-state probability vector $\pi = (\pi_{i,j}), i, j \in E$ of the system states in the level-wise matrix-geometric form [11] (for more details on the method see e.g. [3,8])

$$\pi_k = \pi_{k-1} R, \quad k \geqslant 1, \tag{4}$$

where $\pi_k = (\pi_{k,1}, \ldots, \pi_{k,m})$, and R is the minimal nonnegative (square matrix of order m) solution of nonlinear matrix equation (1), provided the stability condition holds [11]

$$\alpha A^{(0)} \mathbf{1} < \alpha \sum_{k=2}^{N} (k-1) A^{(k)} \mathbf{1}, \tag{5}$$

where the stochastic vector α is the solution of the linear system

$$\begin{cases} \alpha A = 0 \\ \alpha \mathbf{1} = 1. \end{cases} \tag{6}$$

It is relatively easy to see (we refer to e.g. [5]) that the matrices R and $A^{(0)}$ have the same rank. Thus, we restrict the analysis to the case $A^{(0)}$ is nonsingular.

We define the generator function

$$G(\xi) = \sum_{i=0}^{N} \xi^i A^{(i)}.$$

It is easy to show that

$$G(\xi) = (\xi I - R) J(\xi, R), \tag{7}$$

where

$$J(\xi, R) := \sum_{i=0}^{N-1} \xi^i E_i, \tag{8}$$

and

$$E_i = \sum_{j=i+1}^{N} R^{j-i-1} A^{(j)}. \tag{9}$$

The derivation of (7) follows the Residual Theorem [9].

It follows from (3) that $G(1)\mathbf{1} = \mathbf{0}$. It is known, that in a stable system the spectrum $\mathrm{sp}(R) < 1$ (i.e. the largest modulus eigenvalue, being simple, real and nonnegative, see e.g. [8]), and $I - R$ is nonsingular (which easily follows from diagonal dominance), hence it follows from (7) that

$$J(1, R)\mathbf{1} = \mathbf{0}. \tag{10}$$

Consider now $J(1, R)$. It is easy to obtain from (8) that

$$J(1, R) = E_0 + K(R), \tag{11}$$

where

$$K(R) := \sum_{i=1}^{N-1} E_i. \tag{12}$$

We conventionally set $0^0 = 1$ in E_0.

Now consider $G(0)$. By definition, $G(0) = A^{(0)}$. However, noting that (8) provides $J(0, R) = E_0$, we obtain from (7) (and also can see directly from (1)) that

$$A^{(0)} = -RE_0. \tag{13}$$

Thus, since $A^{(0)}$ is nonsingular, then E_0 is nonsingular. Moreover, it follows from (9) that $-E_0 = -A^{(1)} - \sum_{i=2}^{N} R^{i-1}A^{(i)}$, that is, $-E_0$ has positive diagonal elements and nonpositive off-diagonal elements. This means that E_0 is the so-called M-matrix [12]. Hence, it is known that $(-E_0)^{-1}$ is by definition a nonnegative matrix (the proof of this M-matrix definition equivalence is given in [12]). Moreover, since $K(R)$ is also nonnegative, then it follows from (10) and (11) that

$$(-E_0)^{-1}K(R)\mathbf{1} = \mathbf{1},$$

that is, the matrix $(-E_0)^{-1}K(R)$ is stochastic. Hence

$$\left|\det\left[(-E_0)^{-1}K(R)\right]\right| \leqslant 1.$$

It now follows from (12) and (9) that

$$K(R) \geqslant \sum_{i=2}^{N} A^{(i)},$$

since for $i = 1, \ldots, N-1$, the matrices $R^{j-i-1}A^{(j)}, j \geqslant i+2$, are nonnegative. Thus, the matrix $(-E_0)^{-1}\sum_{i=2}^{N} A^{(i)}$ is nonnegative and substochastic, which provides

$$\left|\det\left[(-E_0)^{-1}\sum_{i=2}^{N} A^{(i)}\right]\right| \leqslant 1.$$

(Note that if X is substochastic, i.e. $X\mathbf{1} \leqslant \mathbf{1}$, then it is easy to show, that $\det X \leqslant 1$ e.g. by considering a stochastic matrix XD, where $D = \operatorname{diag}(d)$ is diagonal matrix with $d \geqslant \mathbf{1}$.) Thus

$$|\det \sum_{i=2}^{N} A^{(i)}| \leqslant |\det E_0|.$$

Recalling (13), it follows that

$$|\det A^{(0)}| = |\det R||\det E_0| \geqslant |\det R|\left|\det \sum_{i=2}^{N} A^{(i)}\right|.$$

We have completed the proof of the following

Theorem 1. *If the minimal nonnegative solution R of (1) is nonsingular, then*

$$|\det R| \leqslant \left| \frac{\det A^{(0)}}{\det \sum_{i=2}^{N} A^{(i)}} \right|. \tag{14}$$

3 Discussion

We stress that the upper bound for absolute value of the determinant $|\det R|$ is given in terms of the matrices $A^{(i)}, i = 0, \ldots, N$ known in advance. Note also that it immediately follows from $\mathrm{sp}(R) < 1$ that $|\det R| < 1$. Thus, for practical purpose, one of these bounds may be used.

Let $\eta = \mathrm{sp}(R)$ be the spectrum of R. Then it is easy to see that $\eta \leqslant |\det R|$, which, together with (14), provides an upper bound for the spectrum. In particular we note that for a classical M/M/1 system, the inequality (14) becomes equality $\rho = \lambda/\mu$, where ρ is the server load, λ is an arrival intensity, and μ is the service rate.

Let now $|\theta|$ be the absolute value of minimal (possibly complex) eigenvalue of rate matrix R. Then $|\det R| \geqslant |\theta|^m$, where, recall, m is the size of square matrix R. Thus it follows from (14) that

$$|\theta| \leqslant \left| \frac{\det A^{(0)}}{\det \sum_{i=2}^{N} A^{(i)}} \right|^{1/m}.$$

This inequality may be used as an additional constraint when searching for θ numerically (some numerical methods of obtaining minimal eigenvalue, however, in M-matrix, can be found in [1]). Finally we note that the marginal probability of a level $i \geqslant 0$, $\pi_i \mathbf{1}$, is known to asymptotically decrease approximately as η^i for i large. Then the upper bound (14) together with $\eta < |\det R|$ may be used to obtain a rough approximation for the level B such that $\pi_B \mathbf{1} < \varepsilon$ for given small constant ε. However, we leave a detailed study of these practical applications for future research.

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